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Matrix models without scaling limit

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Abstract

In the context of hermitean one-matrix models we show that the emergence of the NLS hierarchy and of its reduction, the KdV hierarchy, is an exact result of the lattice characterizing the matrix model. Said otherwise, we are not obliged to take a continuum limit to find these hierarchies. We interpret this result as an indication of the topological nature of them. We discuss the topological field theories associated with both and discuss the connection with topological field theories coupled to topological gravity already studied in the literature.

1 Introduction

One-matrix models give rise to a lattice hierarchy of (differential-difference) equations, as well as to a discrete string equation. In the usual approach [1], [2], one takes a continuum limit and recovers (in the even potential case) the KdV hierarchy and the continuum string equation. These two objects can be interpreted both in terms of a topological field theory and in terms of intersection theory of moduli space of Riemann surfaces [3],[4],[5],[6],[7].

In the present paper we show that both the KdV hierarchy and the corresponding string equation are intrinsic to the matrix model lattice—they are not simply a result of an accidentally lucky scaling limit of the lattice, as oral tradition seems sometimes to put it. We do it by extracting the KdV hierarchy and string equations directly from the lattice counterparts without passing through a limiting procedure¹. The method consists essentially in using the first flow equation to eliminate the difference operations in the remaining equations. The latter become a hierarchy of differential equations.

This result is not surprising after all. Let us clarify our point of view with the help of an analogy: the Euler number of a Riemann surface Σ of genus h can be calculated with a continuum procedure by means of

$$2 - 2h = \frac{1}{4\pi} \int_{\Sigma} \sqrt{g} R \quad (1)$$

where R is the scalar curvature w.r.t. the metric g ; but it can also be calculated by means of

$$2 - 2h = V - L + F \quad (2)$$

with reference to a simplicial approximation of Σ , in which V, L, F are the number of vertices, links and faces, respectively.

The KdV hierarchy and string equation obtained via a lattice scaling limit are akin to the RHS of eq.(1). The KdV hierarchy and string equation we discuss in this paper are parallel to the RHS of eq.(2). Since KdV hierarchy and corresponding string equation have a topological significance, the analogy just presented is more than a simple suggestion. The same can be

Actually in this paper we do more. We study the complete one-matrix model (not simply the even potential case), we define the corresponding lattice hierarchy (the Toda lattice hierarchy), and, with the method of [8], we map it to a differential hierarchy, the non linear Schrödinger (NLS) hierarchy. The above mentioned KdV hierarchy is

¹However, since in the continuum limit we may have to renormalize and relabel our objects, the two approaches are expected to differ by renormalization or relabeling of the coupling constants.

nothing but a reduction of this hierarchy. Finally we discuss the topological field theory associated to NLS. We show that it belongs to the family of topological conformal models with two primaries coupled to topological gravity studied in [3],[5].

The paper is organized as follows. In section 2 we review the method presented in [8] to extract differential hierarchies from the Toda lattice hierarchy. In section 3 we review and complete the reduction to the NLS hierarchy and the corresponding string equation, which corresponds to the most general one-matrix model. In section 4 we examine the even potential case, i.e. we show how to extract the KdV hierarchy from the NLS hierarchy and the corresponding string equation. We also discuss the corresponding well-known topological field theory. In section 5 we propose a topological field theory corresponding to the most general one-matrix model. It turns out to correspond to the topological theory mentioned above.

2 Review: from Toda lattice to differential hierarchies

This section is a review of [8]. First some notations. Given a matrix M , we will denote by M_- the strictly lower triangular part and by M_+ the upper triangular part including the main diagonal. We deal with $\infty \times \infty$ matrices. In due time we will pass to semi-infinite matrices. As usual E_{ij} denotes the matrix $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We also use

$$I_{\pm} \equiv \sum_{i=-\infty}^{\infty} E_{i,i\pm 1}, \quad \rho = \sum_{i=-\infty}^{\infty} iE_{ii}$$

Throughout this paper λ denotes the spectral parameter, and Λ represents an infinite dimensional column vector with components $\Lambda_n = \lambda^n$, $n \in \mathbf{Z}$. The vector Λ is our elementary starting point. From it we obtain η

$$\eta = \exp\left(\sum_{r=1}^{\infty} t_r \lambda^r\right) \Lambda \quad (3)$$

In (3) t_r are time or flow parameters. On η one can naturally define a (elementary) linear system

$$\begin{aligned} \lambda \eta &= \partial \eta = I_+ \eta \\ \lambda^r \eta &= \frac{\partial}{\partial t_r} \eta = \partial^r \eta = I_+^r \eta \\ \frac{\partial}{\partial \lambda} \eta &= P_0 \eta, \quad P_0 = \rho I_- + \sum_{r=1}^{\infty} r t_r I_+^{r-1} \end{aligned} \quad (4)$$

Throughout the paper ∂ denotes the derivative $\frac{\partial}{\partial t_1}$. It is easy to see that the spectral and flow equations are automatically compatible.

Next we introduce the “wave matrix” W :

$$W = 1 + \sum_{i=1}^{\infty} \sum_{n=-\infty}^{+\infty} w_i(n) E_{n,n-i} \quad (5)$$

where $w_i(n)$ are functions of the time parameters. We impose them to be determined by the equations of motion

$$\frac{\partial}{\partial t_r} W = Q_+^r W - W I_+^r, \quad Q = W I_+ W^{-1} \quad (6)$$

Finally we introduce the vector $\Psi = W\eta$. In terms of all these objects the dynamical system we have defined can be written as

$$\begin{aligned} Q\Psi &= \lambda\Psi \\ \frac{\partial}{\partial t_r} \Psi &= Q_+^r \Psi \\ \frac{\partial}{\partial \lambda} \Psi &= P\Psi, \quad P = W P_0 W^{-1} \end{aligned} \quad (7)$$

The compatibility conditions of this linear system form the so-called *discrete* KP-hierarchy (Toda lattice hierarchy)

$$\frac{\partial}{\partial t_r} Q = [Q_+^r, Q] \quad (8)$$

together with the *trivial* relation

$$[Q, P] = 1 \quad (9)$$

We have

$$Q = I_+ + \sum_{i=0}^{\infty} \sum_{n=-\infty}^{+\infty} a_i(n) E_{n,n-i}, \quad P = \sum_{r=1}^{\infty} r t_r Q^{r-1} + \sum_{i=0}^{\infty} v_i Q^{-i-1} \quad (10)$$

where v_i are diagonal matrices. The $a_i(n)$ and $w_i(n)$ are different coordinates of the system.

The Toda lattice hierarchy (8) consists of an infinite set of differential-difference equations. In the rest of this section we review the procedure to pass from the above matrix formalism to a related (pseudo-differential) operator formalism and obtain a new hierarchy which consists merely of differential equations. This procedure is referred to

above as the mapping from the discrete hierarchy to the corresponding differentiable hierarchy.

The essential ingredient is the remark that equation (3) implies

$$\eta_n = \partial^{n-m} \eta_m, \quad \forall n, m : \text{integers}$$

This leads to

$$\Psi_n = (W\eta)_n = \sum_{i=-\infty}^n W_{ni} \eta_i = \sum_{i=-\infty}^n W_{ni} \partial^{i-n} \eta_n = \hat{W}_n \eta_n \quad (11)$$

where

$$\hat{W}_n = 1 + \sum_{i < n} W_{ni} \partial^{i-n} = 1 + \sum_{i=1}^{\infty} w_i(n) \partial^{-i} \quad (12)$$

i.e. the “wave matrix” W can be considered as an infinite diagonal matrix, whose components are invertible differential operators. Using this fact we can transform the linear system (7) into the differential linear system

$$\begin{aligned} \hat{Q}_n \Psi_n &= \lambda \Psi_n \\ \frac{\partial}{\partial t_r} \Psi_n &= (\hat{Q}_n^r)_+ \Psi_n, \quad \forall n \text{ integer} \\ \frac{\partial}{\partial \lambda} \Psi_n &= \left(\sum_{r=1}^{\infty} r t_r \hat{Q}_n^{r-1} + \sum_{i=0}^{\infty} v_i(n) \hat{Q}_n^{-i-1} \right) \Psi_n \equiv \hat{P}_n \Psi_n \end{aligned} \quad (13)$$

where the subscript “+” selects now the non-negative powers of the derivative ∂ , and

$$\hat{Q}_n = \hat{W}_n \partial \hat{W}_n^{-1} = \partial + \sum_{i=1}^{\infty} u_i(n) \partial^{-i} \quad \forall n \text{ integer} \quad (14)$$

The compatibility conditions are

$$\frac{\partial}{\partial t_r} \hat{Q}_n = [(\hat{Q}_n^r)_+, \hat{Q}_n] \quad (15)$$

beside the trivial one

$$[\hat{Q}_n, \hat{P}_n] = 1 \quad (16)$$

Eq.(15) specifies the *differential* hierarchy promised above. Actually we have an infinite number of hierarchies, one for each integer n . However they are not independent as they are related by the t_1 flow; we will see below that in the particular case of one-matrix models all these hierarchies are isomorphic.

3 Reductions corresponding to one-matrix models. The NLS hierarchy.

It is well known [11] that the data of a one-matrix model can be encoded in a corresponding discrete linear system. Here we refer to hermitean one-matrix models, specified by an $N \times N$ hermitean matrix M , with partition function and potential given by

$$Z_N(t) = \int dM e^{Tr V(M)}, \quad V(M) = \sum_{r=1}^{\infty} t_r M^r$$

Henceforth we will consider reductions of the large system introduced in the previous section so that they reproduce systems corresponding to these one-matrix models. Our purpose is to obtain the three crucial ingredients of matrix models: the identification of the partition function, the hierarchy of differential equations that give us the flows with respect to the coupling parameters and the string equation.

First of all we reduce the system to semi-infinite matrices. In other words the index n in eqs.(5,10) will run from now on from 0 up to $+\infty$. With this proviso the case relevant to one-matrix models is specified by the conditions

$$a_0(n) = S_n, \quad a_1(n) = R_n, \quad a_i(n) = 0, \quad \forall n \geq 2$$

The discrete t_1 flow gives

$$\frac{\partial}{\partial t_1} \psi_n = \psi_{n+1} + S_n \psi_n \tag{17}$$

$$\frac{\partial}{\partial t_1} S_n = R_{n+1} - R_n \tag{18}$$

$$\frac{\partial}{\partial t_1} R_n = R_n (S_n - S_{n-1}) \tag{19}$$

while the spectral equation (see (7)) writes

$$\lambda \psi_n = \psi_{n+1} + S_n \psi_n + R_n \psi_{n-1} = \partial \psi_n + R_n \frac{1}{\partial - S_{n-1}} \psi_n$$

due to (17).

Therefore, in this case, the pseudo-differential operator \hat{Q}_n is

$$\hat{Q}_n = \partial + R_n \frac{1}{\partial - S_{n-1}}$$

Now (19) gives

$$S_{n-1} = S_n - \frac{R'_n}{R_n}$$

and it is easy to prove that

$$\hat{Q}_n = \partial + \sum_{l=1} u_l(n) \partial^{-l} \quad (20)$$

$$u_l(n) = (-\partial + S_n)^{l-1} R_n \quad (21)$$

In these equations a prime denotes derivative with respect to t_1 . As anticipated in the previous section the $u_l(n)$'s depend only on the coordinates S_n and R_n and their derivatives. As a consequence the infinitely many KP-hierarchies we noticed in the previous section are all isomorphic.

At this point the analysis is universal, i.e. it does not depend on the label n , therefore we simply omit it, and denote $R_n(t_1)$, $S_n(t_1)$ by $R(t_1)$ and $S(t_1)$, respectively.

We construct a bi-Hamiltonian structure for this model as follows. First we define

$$Tr(A) = \int dx a_{-1}(x), \quad A = \dots + a_{-1}(x) \partial^{-1} + \dots$$

for any pseudo-differential operator A . Then for any differential operator X we set $f_X(\hat{Q}) = \langle \hat{Q} X \rangle$. Then we define

$$\begin{aligned} \{f_X, f_Y\}_1(\hat{Q}) &= \langle \hat{Q}[Y, X] \rangle \\ \{f_X, f_Y\}_2(\hat{Q}) &= \langle (X\hat{Q})_+ Y \hat{Q} \rangle - \langle (\hat{Q}_X)_+ \hat{Q}_Y \rangle \end{aligned} \quad (22)$$

The following quantities are in involution

$$H_k = \frac{1}{k} Tr(\hat{Q}^k), \quad \forall k \geq 1$$

and the two Poisson brackets are compatible

$$\{H_{r+1}, f\}_1 = \{H_r, f\}_2 \quad \text{for any function } f \quad (23)$$

On the coordinates R and S the above Poisson brackets are

$$\begin{aligned} \{R(x), R(y)\}_1 &= 0 & \{S(x), S(y)\}_1 &= 0 \\ \{R(x), S(y)\}_1 &= -\partial_x \delta(x - y) \end{aligned}$$

and

$$\{R(x), R(y)\}_2 = (2R(x)\partial_x + R'(x))\delta(x-y) \quad (24)$$

$$\{S(x), S(y)\}_2 = 2\partial_x\delta(x-y) \quad (25)$$

$$\{R(x), S(y)\}_2 = (S(x)\partial_x - \partial_x^2)\delta(x-y) \quad (26)$$

Remark. For later use we notice that, upon adding the term

$$+2 \int [\hat{Q}, Y]_{(-1)} \left(\partial^{-1} [\hat{Q}, X] \right)$$

to the RHS of(23), we would obtain

$$\{S(x), S(y)\}_2 = 0 \quad (27)$$

instead of (25), while the other two brackets would not change. However in this case the above compatibility condition of the two Poisson brackets would not hold as it stands. It has been shown recently that such a modified second Poisson structure appears in at least two other situations: in the $SL(2, \mathbb{R})$ WZW model[10], and in the conformal affine Liouville theory[9]. It is known to be connected with a W_∞ structure.

Let us come now to the flow equations. They can be expressed in compact form using the two polynomials

$$F_r(x) = \frac{\delta H_r}{\delta S(x)} \quad G_r(x) = \frac{\delta H_r}{\delta R(x)} \quad \forall r \geq 1$$

In particular, $F_1 = 0$ and $G_1 = 1$. They satisfy the recursion relations

$$F'_{r+1} = SF'_r - F''_r + 2RG'_r + R'G_r \quad (28)$$

$$G'_{r+1} = 2F'_r + G''_r + (SG_r)' \quad (29)$$

and in terms of them

$$\frac{\partial}{\partial t_r} S = G'_{r+1}, \quad \frac{\partial}{\partial t_r} R = F'_{r+1} \quad (30)$$

This is the hierarchy characterizing the one-matrix models. We see immediately that this is the NLS hierarchy. In fact the t_2 flow equations are:

$$\frac{\partial S}{\partial t_2} = S'' + 2S'S + 2R', \quad \frac{\partial R}{\partial t_2} = -R'' + 2(RS)' \quad (31)$$

If we change variables as follows

$$S = (\ln \psi)', \quad R = -\psi \bar{\psi}$$

and denote by a dot the derivative with respect to t_2 , eqs.(31) become

$$\dot{\psi} = \psi'' - 2\psi^2\bar{\psi}, \quad \dot{\bar{\psi}} = -\bar{\psi}'' + 2\bar{\psi}^2\psi$$

which contain as a particular case the non-linear Schrödinger equation. For this reason we will henceforth refer to this hierarchy as the NLS one.

Next we establish the connection of the matrix model partition function with the coordinates of our system as follows. In the discrete case, we have

$$\frac{\partial}{\partial t_1} \ln Z_N(t) = \sum_{i=0}^{N-1} S_i \quad (32)$$

Using the first flow

$$\frac{\partial}{\partial t_1} S_n = R_{n+1} - R_n$$

we obtain

$$\frac{\partial^2}{\partial t_1^2} \ln Z_N(t) = R_N \quad (33)$$

We will use R_N as the definition of the “specific heat”. We know that, like any other R_n , it satisfies the hierarchy (30) of equations, so we may simply drop the label N.

Finally, to completely characterize the systems corresponding to one-matrix models, we need the string equation. The equation

$$[\hat{Q}_n, \hat{P}_n] = 1 \quad (34)$$

does not imply any restriction on the model as long as P has the general form (15). One-matrix models are characterized by the following restriction on the form of P

$$(\hat{P}_n)_- = 0, \quad \text{i.e.} \quad \hat{P}_n = \sum_{r=1}^{\infty} r t_r (\hat{Q}_n^{r-1})_+ \quad (35)$$

where the label “+” denotes the pure differential part and “−” the remaining part of a pseudodifferential operator. By *string equation* we mean (34) together with the condition (35). Inserting the flow equations in the string equation we obtain the Virasoro constraints [11].

$$L_n Z_N = 0, \quad n \geq -1 \quad (36)$$

where

$$\begin{aligned}
L_{-1} &= \sum_{r=2}^{\infty} r t_r \frac{\partial}{\partial t_{r-1}} + N t_1 \\
L_0 &= \sum_{r=1}^{\infty} r t_r \frac{\partial}{\partial t_r} + N^2 \\
L_n &= \sum_{r=1}^{\infty} r t_r \frac{\partial}{\partial t_{r+n}} + 2N \frac{\partial}{\partial t_n} + \sum_{r=1}^{n-1} \frac{\partial^2}{\partial t_r \partial t_{n-r}}, \quad n > 0
\end{aligned} \tag{37}$$

4 The KdV hierarchy and the corresponding TFT.

The KdV hierarchy in the context of matrix models has been repeatedly analyzed, and our results agree with the literature. However it is interesting to reconsider the problem from our point of view, both in itself and as a preparation for the next section.

The KdV hierarchy and the corresponding string equation are obtained by further restricting the system of the previous section. This is achieved in the following way. Let us set $S = 0$ in (30) and discard the t_{2r} flows. Then from eqs.(28,29) we get

$$G_{2r} = 0, \tag{38}$$

$$G'_{2r+1} = 2F'_{2r} \tag{39}$$

$$F'_{2r+1} = -F''_{2r} \tag{40}$$

and

$$\frac{\partial R}{\partial t_{2r+1}} = F'_{2r+2} = (\partial^3 + 4R\partial + 2R')F_{2r} \tag{41}$$

where the initial condition $F_2 = R$ has been used. Eq.(41) is the recursion relation for the KdV hierarchy. In particular we have

$$\frac{\partial}{\partial t_3} R = R''' + 6RR' \tag{42}$$

The connection with the partition function is the same as in the above section, eq.(33). This means in particular that the reduced partition function, Z , is a τ -function of the KdV hierarchy.

It remains for us to find the appropriate form of the Virasoro constraints. Let us start with the first Virasoro constraint. It was proven in [8] that we can put the string equation in the form

$$\sum_{r=2}^{\infty} r t_r G'_r + 1 = 0 \tag{43}$$

Using (38,39) we find

$$\sum_{r=1}^{\infty} (2r+1)t_{2r+1} \frac{\partial R}{\partial t_{2r-1}} + \frac{1}{2} = 0 \quad (44)$$

Essentially from this condition we can obtain the following Virasoro constraints (see the Appendix for the explicit derivation and for the problems connected with it)

$$\mathcal{L}_n Z = 0, \quad n \geq -1 \quad (45)$$

where

$$\begin{aligned} \mathcal{L}_{-1} &= \sum_{k=1}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2k-1}} + \frac{t_1^2}{8} \\ \mathcal{L}_0 &= \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2k+1}} + \frac{1}{16} \\ \mathcal{L}_n &= \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2k+2n+1}} + \frac{1}{2} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial t_{2k+1} \partial t_{2n-2k-1}}, \quad n > 0 \end{aligned} \quad (46)$$

Now all the tools we need are at hand.

It has been shown by Witten [4] and Kontsevitch [6] that the emergence of the KdV hierarchy and the string equation have a definite topological meaning. In particular, considered in the framework of a topological field theory coupled to 2D gravity they constitute a very efficient method to calculate correlation functions. Following [4] we define the action corresponding to the KdV system as

$$S = S_0 - \sum_{k=0}^{\infty} t_{2k+1} \int \sigma_{2k+1}^{(2)} \quad (47)$$

The corresponding free energy is

$$\mathcal{F} = \sum_{\{n_i\}} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!} \dots \frac{t_k^{n_k}}{n_k!} \langle \sigma_1^{n_0} \sigma_3^{n_1} \dots \sigma_{2k+1}^{n_k} \rangle_0 \quad (48)$$

where the sum extends over all the n-tuples of integers $\{n_i\} = \{n_1, n_2, \dots, n_k\}$, and $\langle \cdot \rangle_0$ stands for a correlation function calculated when all the couplings vanish. Let us also introduce the notation

$$\frac{\partial^{n_1}}{\partial t_1^{n_1}} \frac{\partial^{n_3}}{\partial t_3^{n_3}} \dots \frac{\partial^{n_k}}{\partial t_{2k+1}^{n_k}} \mathcal{F} \equiv \ll \sigma_1^{n_1} \sigma_2^{n_2} \dots \sigma_{2k+1}^{n_k} \gg$$

If we identify this free energy with the one introduced above, $\mathcal{F} = \ln Z$, then we have $R = \mathcal{F}'' = \ll PP \gg$, where $P \equiv \sigma_1$, we can use the KdV hierarchy and string equation to calculate the correlation functions for any genus. The calculation is particularly easy in genus 0 and we will limit ourselves to this case. Then the full KdV hierarchy can be replaced by the dispersionless KdV hierarchy

$$\frac{\partial}{\partial t_{2k+1}} R = 2^k \frac{(2k+1)!!}{k!} R^k R' \quad (49)$$

This can be easily obtained by rescaling t_k and R by suitable powers of N and keeping the dominant contribution in $\frac{1}{N}$, which simply amounts to discarding all the higher derivatives in t_1 in the flow equations. A straightforward result is then the following

$$\ll \sigma_{2k_1+1} \sigma_{2k_2+1} \dots \sigma_{2k_n+1} \gg = \partial^{-2} \frac{\partial^n}{\partial t_1^n} \prod_{i=1}^n \frac{2^{k_i} (2k_i+1)!!}{k_i!} \frac{R^{k_1+k_2+\dots+k_n+1}}{k_1+k_2+\dots+k_n+1} \quad (50)$$

where, in the RHS, ∂^{-1} denotes here true integration with respect to t_1 (true integration as opposed to formal integration, since here we cannot exclude a priori non trivial integration constants). The expressions of the above correlation functions in terms of the coupling constants can be found by solving the Landau-Ginzburg type equation which is obtained by differentiating the \mathcal{L}_{-1} Virasoro condition

$$\frac{\partial \mathcal{F}}{\partial t_1} = 2 \sum_{k=1}^{\infty} (2k+1) t_{2k+1} \frac{\partial \mathcal{F}}{\partial t_{2k-1}} + \frac{t_1^2}{2} \quad (51)$$

where we have shifted $t_3 \rightarrow t_3 - \frac{1}{6}$. For example the first critical point is met at $t_{2k+1} = 0$ for $k > 0$ (small phase space). In this case we have

$$\langle P \rangle = \frac{t_1^2}{2}, \quad \langle PP \rangle = R = t_1, \quad \langle PPP \rangle = 1$$

$\langle \cdot \rangle$ denotes a correlation function in the small phase space. Therefore if we replace R with t_1 in (50) we get immediately the correlation functions $\langle \sigma_{2k_1+1} \sigma_{2k_2+1} \dots \sigma_{2k_n+1} \rangle$ in the small phase space, up to arbitrary constants coming from double integration with respect to t_1 . These integration constants are determined by the string equation. For example, from the \mathcal{L}_n Virasoro condition (46), after extracting the part relevant to genus 0 and shifting t_3 as above, we get the recursive relation

$$\langle \sigma_{n+3} \rangle = 2t_1 \langle \sigma_{2n+1} \rangle + \sum_{k=0}^{n-1} \langle \sigma_{2k+1} \rangle \langle \sigma_{2n-2k-1} \rangle$$

Similar recursive relations for multiple correlation functions can be obtained by first differentiating the \mathcal{L}_n constraints with respect to the couplings and then repeating the above derivation. One can verify that all these recursive relations are satisfied by

$$\langle \sigma_{2k_1+1} \sigma_{2k_2+1} \dots \sigma_{2k_n+1} \rangle = \frac{\partial^{n-2}}{\partial t_1^{n-2}} \prod_{i=1}^n \frac{2^{k_i} (2k_i + 1)!!}{k_i!} \frac{t_1^{k_1+k_2+\dots+k_n+1}}{k_1 + k_2 + \dots + k_n + 1} \quad (52)$$

In the case $n = 1$ the first symbol in the RHS has to be understood as formal integration w.r.t. t_1 (i.e. integration with vanishing integration constant – we will see in the next section that this is not always the case). The normalization we used for the σ_{2k+1} is completely natural in the context we presented, but leads to results which differ in normalization from the ones in the literature. However it is enough to define new couplings t_r

$$t_r = 2^r \frac{(2r + 1)!!}{r!} t_{2r+1}, \quad r = 0, 1, 2, \dots \quad (53)$$

to recover the results of Gross and Migdal [1].

Two comments are in order to end this section. What we have done so far tells us that the correlation functions of the fields σ_k can be calculated by using the path integral of one-matrix models as correlation functions of $Tr(M^{2k+1})$. In other words we have the correspondence

$$Tr(M^{2k+1}) \leftrightarrow \sigma_{2k+1}$$

The second remark is connected with the first. One should not confuse the fact that in one-matrix models the conditions $S_n = 0$ and $t_{2k+1} = 0$ are related with the fact that we put $S = 0$ at the beginning of this section. What we did is the following: the most general one-matrix model allowed us to write down an integrable system (with $S \neq 0$); next we defined a consistent reduction of the system specified by the condition $S = 0$; the reduced theory depends on the odd coupling constants. One should bear in mind that the constraint $S = 0$ is meant to be applied to the integrable system, not to the one-matrix model path integral.

5 The NLS hierarchy and the corresponding TFT.

In this section we want to do for the larger NLS hierarchy what we have done in the previous section for the KdV hierarchy; i.e. we try to define a topological field theory coupled to gravity that corresponds to it.

One quickly realizes that it is not possible to represent this theory simply by means of the puncture operator P and its descendants as in the KdV case. There must be

another field, which we call Q , coupled through some parameter in the theory. Since the only parameter in the theory not already associated to some field is the size N of the lattice, we assume that Q is coupled to the theory with coupling N^2 . Of course we have to define the differentiation with respect to N . We do it as follows: for any function f_N on the lattice we define the derivative ∂_0 by means of

$$(e^{\partial_0} - 1) f_N \equiv f_{N+1} - f_N \quad (54)$$

Setting $D_0 = e^{\partial_0} - 1$ we have, conversely,

$$\partial_0 = \ln(D_0 + 1) \quad (55)$$

One can easily verify that ∂_0 can be identified with the derivative with respect to N ³.

In particular, using the first flow equations, (19, 18), we obtain

$$(e^{\partial_0} - 1) R_N = R_{N+1} - R_N = S'_N \quad (56)$$

$$(e^{\partial_0} - 1) S_N = S_{N+1} - S_N = (\ln(R_N + S'_N))' \quad (57)$$

Next, R_N is connected to the partition function through eq.(33). The relation of S_N to the partition function is found by applying D_0 to both sides of eq.(32). We obtain

$$(e^{\partial_0} - 1) \frac{\partial}{\partial t_1} \ln Z_N = S_N \quad (58)$$

Applying D_0 and $\frac{\partial}{\partial t_1}$ to eqs.(33, 58), we obtain identities, except when we apply D_0 to (58). In that case we obtain the following compatibility condition

$$(e^{\partial_0} - 1)^2 \ln Z_N = \ln(R_N + S'_N) + f(N) \quad (59)$$

where we have integrated once with respect to t_1 and $f(N)$ is the corresponding arbitrary (model dependent) integration constant.

Eqs.(56,57,33,58) and the compatibility condition (59), together with the NLS flows (30) and the Virasoro constraints (36), are the basis of our subsequent discussion. Henceforth, for the sake of uniformity, we relabel the new coupling N as t_0 . Therefore

$$t_0 = N, \quad \partial_0 = \frac{\partial}{\partial t_0}, \quad \text{etc.}$$

²Recently a similar suggestion was made in ref.[15]. We think the spirit of this paper is close to ours, even though the methods employed are different – see also [16].

³It is obvious that when differentiating with respect to N we understand a continuous extension of the integer parameter N .

Moreover we drop the label N from R, S and $\ln Z$ and understand that it is included as t_0 in the collective label t which represent the coupling constants. In particular the first Virasoro constraint can be rewritten

$$\sum_{r=2}^{\infty} r t_r \frac{\partial \mathcal{F}}{\partial t_{r-1}} - \frac{\partial \mathcal{F}}{\partial t_1} + t_0 t_1 = 0 \quad (60)$$

where $\mathcal{F}(t) = \ln Z(t)$ and, for later purposes, we have shifted t_2 by $-\frac{1}{2}$ (which sets the first critical point at $t_2 = 0$).

The result of adding the new coupling t_0 can be regarded as an enlargement of the hierarchy (30). Beside the t_k flows with $k \geq 2$ we have now the t_0 flow as well. This is obtained by inverting eqs.(56) and (57)

$$\partial_0 R = S' - \frac{1}{2} (\ln(R + S'))'' + \dots \quad (61)$$

$$\partial_0 S = (\ln(R + S'))' - \frac{1}{2} \left\{ \ln \left(1 + \frac{S'(\ln(R + S'))'}{R + S'} \right) \right\}' + \dots \quad (62)$$

where dots denote higher order derivatives in t_1 .

Let us now pass to the field theory language. We want the free energy \mathcal{F} to be generated by an action

$$S = S_0 - \sum_{k=0}^{\infty} t_k \int \sigma_k^{(2)} \quad (63)$$

The problem is now to express the (perturbed) correlation functions

$$\frac{\partial^{n_0}}{\partial t_0^{n_0}} \frac{\partial^{n_1}}{\partial t_1^{n_1}} \dots \frac{\partial^{n_k}}{\partial t_k^{n_k}} \mathcal{F} \equiv \ll \sigma_0^{n_0} \sigma_1^{n_1} \dots \sigma_k^{n_k} \gg$$

in terms of R and S . Here

$$\sigma_0 = Q, \quad \sigma_1 = P$$

So, in particular,

$$R = \ll PP \gg, \quad S = \ll (e^Q - 1)P \gg \quad (64)$$

and the compatibility condition (59) means

$$\ll (e^Q - 1)^2 \gg = \ln \left(\ll PPe^Q \gg \right) + f(t_0) \quad (65)$$

The tools to calculate the correlation functions are the NLS flows and the string equation, or, equivalently, the Virasoro constraints (36) which contain both. Let us draw first some conclusions concerning the small phase space, i.e. the space of couplings when all the t_k are set equal to zero except t_0 and t_1 . The first Virasoro constraint (60) becomes

$$\langle P \rangle = t_0 t_1$$

Therefore

$$\langle PQ \rangle = t_1, \quad \langle PP \rangle = t_0 \quad (66)$$

and, as a consequence,

$$\langle PPQ \rangle = 1 \quad (67)$$

while all the other correlation functions containing at least one P insertion vanish. The correlation functions with only Q insertions depend on the arbitrary function $f(t_0)$ through (65).

So far the correlation functions look the same as those of the \mathbf{CP}^1 model studied by Witten, [3],[5]. We will comment later on this connection. Using the flows (30) we can write down equations for

$$\ll \sigma_{n_1} \dots \sigma_{n_k} Q \gg \quad \text{or} \quad \ll \sigma_{n_1} \dots \sigma_{n_k} P \gg$$

in terms of R, S and their derivatives. In order to obtain

$$\ll \sigma_{n_1} \dots \sigma_{n_k} \gg$$

we have simply to integrate either the first expression above with respect to t_0 or the second with respect to t_1 . The integration constants have to be determined in such a way as to satisfy the string equation.

Let us consider, as an example, the one point correlation functions. The n -th Virasoro constraint can be rewritten, in the small phase space, as

$$\langle \sigma_{n+2} \rangle = t_1 \langle \sigma_{n+1} \rangle + 2t_0 \langle \sigma_n \rangle + \sum_{k=1}^{n-1} (\langle \sigma_k \sigma_{n-k} \rangle + \langle \sigma_k \rangle \langle \sigma_{n-k} \rangle) \quad (68)$$

The two-point functions can be obtained by differentiating the Virasoro constraints with respect to t_k , and so on. In this way we obtain a full set of recursive relations that allow us to calculate all the correlation functions.

The calculation is particularly easy in genus 0 and from now on we limit ourselves to this case. In order to obtain the equations relevant to genus 0 we rescale all the quantities by suitable powers of N .

$$t_k \rightarrow N^{\frac{2-k}{2}} t_k, \quad R \rightarrow NR, \quad S \rightarrow N^{\frac{1}{2}} S$$

and keep the leading terms in $\frac{1}{N}$. For example

$$S = \ll PQ \gg + \sum_{l=1}^{\infty} \frac{1}{l! N^{l+1}} \ll PQ^l \gg$$

In conclusion in genus 0 we have

$$R = \ll PP \gg, \quad S = \ll PQ \gg \quad (69)$$

while the compatibility condition (65) becomes

$$\ll QQ \gg = \ln \ll PP \gg + f_0(t_0) \quad (70)$$

where $f_0(t_0)$ is the appropriate genus zero term, derived from $f(t_0)$ (it is model dependent). The hierarchy (30) becomes the dispersionless hierarchy, i.e.

$$\frac{\partial S}{\partial t_r} = G'_{r+1} = \sum_{\substack{k \\ 0 \leq 2k \leq r}} \binom{r}{2k} \binom{2k}{k} (R^k S^{r-2k})' \quad (71)$$

$$\frac{\partial R}{\partial t_r} = F'_{r+1} = \sum_{\substack{k \\ 2 \leq 2k \leq r+1}} \binom{r}{2k-1} \binom{2k-1}{k} (R^k S^{r-2k+1})' \quad (72)$$

Notice that setting $S = 0$ and keeping only the odd flows and using the redefinitions of the previous section, we obtain (49).

From (71) we obtain

$$\ll \sigma_r Q \gg = \sum_{\substack{k \\ 0 \leq 2k \leq r}} \frac{r!}{(r-2k)!(k!)^2} (R^k S^{r-2k}) \quad (73)$$

while from (72) we get

$$\ll \sigma_r P \gg = \sum_{\substack{k \\ 2 \leq 2k \leq r+1}} \frac{r!}{(r-2k+2)!(k-1)!k!} R^k S^{r-2k+2} \quad (74)$$

What we said so far for genus 0 is valid in the large phase space. Now let us come to the the small phase space. To begin with we have

$$\langle QQ \rangle = \ln t_0 + f_0(t_0)$$

and

$$R = t_0, \quad S = t_1$$

If we insert this into (73) and integrate over t_0 (with a vanishing integration constant), we obtain, in the small phase space

$$\langle \sigma_r \rangle = \sum_{\substack{k \\ 2 \leq 2k \leq r+2}} \frac{r!}{(r-2k+2)!(k-1)!k!} t_0^k t_1^{r-2k+2} \quad (75)$$

We can obtain the same result by integrating (74) with respect to t_1 , but in this case we have to add a suitable t_0 -dependent integration constant.

Notice that, both here and in (73,74), we have made a choice for the integration constants. This choice can be justified on the basis of the string equation. For genus 0 eq.(68) takes the form

$$\langle \sigma_{n+2} \rangle = t_1 \langle \sigma_{n+1} \rangle + 2t_0 \langle \sigma_n \rangle + \sum_{k=1}^{n-1} \langle \sigma_k \rangle \langle \sigma_{n-k} \rangle \quad (76)$$

One can verify that eq.(75) does satisfy (76). In a similar way we can derive multi-point correlation functions.

Let us return to eqs.(66, 67). If we interpret P and Q as primary fields of an underlying topological conformal field theory, these equations give us the metric:

$$\eta_{PQ} = \eta_{QP} = 1, \quad \eta_{QQ} = \eta_{PP} = 0 \quad (77)$$

However a second formulation of the NLS TFT is possible in which Q and P exchange their role, i.e. Q couples to t_1 and P to t_0 . In this case instead of (66, 67) we would have

$$\langle PQ \rangle = t_1, \quad \langle QQ \rangle = t_0, \quad \langle QQP \rangle = 1 \quad (78)$$

This would give us a diagonal metric

$$\eta_{PQ} = \eta_{QP} = 0, \quad \eta_{QQ} = 1 \quad (79)$$

while η_{PP} is some non-vanishing positive number. We will refer to this theory as the second formulation of the NLS TFT, while the previous one will be called the first formulation. For the second formulation we can of course repeat the calculations we carried out above for the first.

It is time now to discuss the nature of the NLS topological field theory. We argue that this field theory belongs to the family of topological conformal models with two primaries coupled to topological gravity, [3]; more precisely it coincides with the explicit version given in [5] of topological \mathbf{CP}^1 sigma model coupled to gravity, up to some modifications. We make the comparison only in genus 0, since the latter is explicitly formulated only in such case. In ref.[5] one can find recursion relations and Virasoro constraints in terms of a function f . We have proved that, if $\langle QQ \rangle = f(\langle PP \rangle) = \ln \langle PP \rangle$, the underlying hierarchy coincides with the dispersionless NLS hierarchy up to rescaling of the coupling constants. So we can identify the model presented in [5] with our first formulation above (with f_0 equal to some constant in (70)), provided we choose the above form of the function f (this is not however the form proposed in [5]).

On the other hand the model of [5] coincides with our second formulation, provided the metric in [5] is chosen in such a way that (78) holds. In this case of course $\langle QQ \rangle = \exp \langle PP \rangle$. The possibility of these two alternative interpretations was envisaged in [3].

Appendix

In this Appendix we derive the Virasoro constraints (45). We start from eq.(43). We can integrate it and obtain

$$\sum_{k=2}^{\infty} k t_k G_k + t_1 = 0 \quad (80)$$

Actually we should not exclude a priori a non-vanishing integration constant in the RHS (see below). However this constant must vanish. One can see it as follows: use the NLS flows and write (80) as

$$\sum_{k=2}^{\infty} k t_k \partial^{-1} \frac{\partial S}{\partial t_{k-1}} + t_1 = 0 \quad (81)$$

Here ∂^{-1} means formal integration (see below). Next notice that this equation is true for $S = S_n$ for any n . Moreover $S_n = \partial \ln h_n$ and $\ln Z_N = \sum_{n=0}^{N-1} \ln h_n + \text{const}$, see [11]. So by summing (80) over n from 0 to $N-1$, we obtain the L_{-1} Virasoro constraint (36) on the lattice. Viceversa, starting from the latter we can just as easily obtain (80).

Now one might be tempted to integrate eq.(80) once more, but this does not make sense in the lattice. Therefore we consider (80) as our starting relation and proceed to the reduction illustrated at the beginning of section 4. We obtain

$$\sum_{k=1}^{\infty} (2k+1)t_{2k+1}\partial^{-1}\frac{\partial \ln Z}{\partial t_{2k-1}} + t_1 = 0 \quad (82)$$

We now integrate this equation with respect to t_1 and obtain

$$\mathcal{L}_{-1}Z = b_{-1}Z \quad (83)$$

which is almost the first Virasoro condition except for the integration constant b_{-1} , where $b_{-1} \equiv b_{-1}(t_3, t_5, \dots)$.

Next we apply the operator

$$\mathcal{D} = \partial^3 + 4R\partial + 2R'$$

to eq.(82). Using (41) and integrating twice, we obtain

$$\mathcal{L}_0Z = (a_0t_1 + b_0)Z \quad (84)$$

where a_0 and b_0 are integration constants depending on t_3, t_5, \dots . Notice that we have included a part of the integration constant $= \frac{1}{16}$ in the definition of \mathcal{L}_0 . We could continue in this way by integrating once (83) and applying \mathcal{D} and so on. We would obtain a series of integration constants a_n and b_n , $n > 0$ similar to the above ones. In order to understand how to calculate them we have to clarify preliminarily a few points.

When considering R and polynomials of R and its derivatives with respect to t_1 we are entitled to use the identities $\partial\partial^{-1} = 1$ and $\partial^{-1}\partial = 1$. In this sense we speak of the symbol ∂^{-1} as a formal integration. The reason is that R and polynomials of R and its derivatives are supposed to belong to a family of objects to which pseudodifferential calculus applies (for example, to the family of functions rapidly decreasing at infinity in the variable t_1). However this is in general not guaranteed for an infinite sum such as, for example, $\sum_{k=2}^{\infty} kt_k G_k$. If we insist on applying ∂^{-1} to such objects as a formal integration, we are bound to run into inconsistencies. Therefore when applying ∂^{-1} to the infinite sums that appear in the string equations we have to interpret it as a true integration and take care of the corresponding integration constants.

On the other hand the KdV hierarchy and its solutions are characterized by homogeneity in t_n with degree assignment

$$\deg(t_n) = n$$

We have consequently

$$\deg(\mathcal{L}_n) = -2n = \deg(a_n t_1 + b_n) \quad (85)$$

This is the first ingredient we will use. The second is the form of the constant a_n and b_n . A rather general form we can imagine for them is a sum of expressions like

$$\sim \prod_{i=1}^s t_{2k_i+1}^{a_i} \quad (86)$$

where a_i are real numbers. We can strongly restrict the enormous number of possibilities implied by equation (86) by using the path integral form of the model we are studying. We remember that in the path integral the terms involving t_n , $n \geq 3$ can be considered as perturbations. Therefore we do not expect anything dramatic to happen when only one of these couplings is set to zero. Thus we can conclude that in (86) all the exponents a_i are non-negative⁴. If we now make a degree analysis of the various possible constants, we conclude that only b_{-1} and b_0 can be non-vanishing, and the latter is a true constant, i.e. it does not depend on t_3, t_5, \dots . What remains for us to do is to calculate these two constants. To this end we can repeat the analysis of the Appendix of [11].

Let us summarize the situation. So far we have found the relations

$$\mathcal{L}_n Z = b_n Z \quad (87)$$

where only b_{-1} and b_0 are non-vanishing. Since

$$[\mathcal{L}_n, \mathcal{L}_m] = (n - m) \mathcal{L}_{n+m}$$

the consistency conditions

$$[\mathcal{L}_n, b_m] - [\mathcal{L}_m, b_n] = (n - m) b_{n+m} \quad (88)$$

must be satisfied. Take these equations for $m = 1$ and $m = 0$. Studying the t_1 dependence we find

$$\frac{\partial b_{-1}}{\partial t_{2k+1}} = 0, \quad \frac{\partial b_0}{\partial t_{2k+1}} = 0, \quad k \geq 0$$

So b_{-1} and b_0 are true constants (for b_0 this was already known independently to us). If we use (88) for $n = 0$ and $m = -1$ and for $n = 1$ and $m = -1$ we immediately conclude that

$$b_{-1} = 0, \quad b_0 = 0$$

⁴This is certainly what happens in the lattice, see (36).

respectively. This complete our proof of (45).

A few comments. Our proof relies on the form (86) for the integration constants and on path integral considerations. This means that we cannot completely exclude the existence of other reductions of the NLS hierarchy to the KdV one (beside the one of section 4), in which the Virasoro conditions (45) are replaced by more complicated conditions (due to non-vanishing integration constants). However we are rather skeptical about this possibility since in the lattice case it was proved without any assumption, [11], that only in the L_{-1} and L_0 Virasoro conditions do non trivial integration constants appear – and are of a very simple form.

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